

# Monotonicity in first-passage percolation

Jean-Baptiste Gouéré \*

## Abstract

We consider standard first-passage percolation on  $\mathbb{Z}^d$ . Let  $e_1$  be the first coordinate vector. Let  $a(n)$  be the expected passage time from the origin to  $ne_1$ . In this short paper, we note that  $a(n)$  is increasing under some strong condition on the support of the law of the passage times on the edges.

## 1 Introduction and results

**First passage percolation.** We consider the graph  $\mathbb{Z}^d$ ,  $d \geq 2$ , obtained by taking  $\mathbb{Z}^d$  as vertex set and by putting an edge between two vertices if the Euclidean distance between them is 1. We consider a family of non-negative i.i.d.r.v.  $\tau = (\tau(e))_{e \in \mathcal{E}}$  indexed by the set of edges  $\mathcal{E}$  of the graph. We interpret  $\tau(e)$  as the time needed to travel along the edge  $e$  (the graph is unoriented).

If  $a$  and  $b$  are two vertices of  $\mathbb{Z}^d$ , we call path from  $a$  to  $b$  any finite sequence of vertices  $r = (a = x_0, \dots, x_k = b)$  such that, for all  $i \in \{0, \dots, k-1\}$ , the vertices  $x_i$  et  $x_{i+1}$  are linked by an edge. We denote by  $\mathcal{C}(a, b)$  the set of such paths. The time needed to travel along a path  $r = (x_0, \dots, x_k)$  is defined by:

$$\tau(r) = \sum_{i=0}^{k-1} \tau(x_i, x_{i+1}).$$

Then, the time needed to go from  $a$  to  $b$  is defined by:

$$T(a, b) = \inf\{\tau(r) : r \in \mathcal{C}(a, b)\}.$$

Let  $e_1, \dots, e_d$  denote the canonical basis vectors of  $\mathbb{R}^d$ . We are interested in the sequence  $(a(n))$  defined by :

$$a(n) = E(T(0, ne_1)).$$

We write  $a'(n)$  for the expected passage times obtained when the paths are restricted to  $\{(x_1, \dots, x_d) : 0 \leq x_1 \leq n\}$ .

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\*Postal address: Université d'Orléans MAPMO B.P. 6759 45067 Orléans Cedex 2 France E-mail: jbgouere@univ-orleans.fr

**Main result and related results.** We denote by  $S_-$  the infimum of the support of the law of the  $\tau(e)$ . We denote by  $S_+$  the supremum of the support.

**Theorem 1** *Assume  $0 < S_-$  and  $S_+ \leq 2S_-$ . Then the sequence  $(a_n)$  is non-decreasing. More precisely, we have:*

$$a(n) \geq a(n-1) + S_- \left[ 1 - \frac{(S_+ - S_-)^2}{S_-^2} \right].$$

Monotonicity of expected passage times seems quite natural and was already conjectured by Hammersley and Welsh in [2]. In [1], Alm and Wierman proved the monotonicity for  $\mathbb{Z} \times \mathbb{N}$  and other 2 dimensional models. In [3], Howard proved the monotonicity for an Euclidean first-passage percolation model. We are not aware of any other positive result.

On the other hand, van den Berg prove in [6] that, when  $d = 2$ , one has  $a'(2) < a'(1)$  when  $\tau(e) = 1$  with small probability and  $\tau(e) = 0$  otherwise. A related result was given by Joshi in [5].

We refer to the review by Howard [4] for a more detailed account.

**Further remarks.**

- The same result holds for the  $a'(n)$ .
- The proof gives that  $T(0, ne_1)$  stochastically dominates the mean of  $n$  dependent copies of  $T(0, (n-1)e_1)$  (see (4) and (1)).

## 2 Proof of Theorem 1

For all  $i$  we consider the following sets of edges:

- $H^i$ : the set of edges  $(x, x + e_1)$  where  $x = (x_1, \dots, x_d)$  is such that  $x_1 = i$ .
- $V^i$ : the set of edges  $(x, x + e_k)$  where  $x_1 = i$  and  $k$  belongs to  $\{2, \dots, d\}$ .

We define new passage times  $\tau^i(e)$  as follows:

- If  $e$  belongs to  $H^i$  then  $\tau^i(e) = 0$ .
- If  $e$  belongs to  $V^i$  then  $\tau^i(e) = +\infty$ .
- Otherwise,  $\tau^i(e) = \tau(e)$ .

We denote by  $T^i(a, b)$  the time needed to travel from  $a$  to  $b$  with the passage times  $\tau^i(e)$ . Note, for all  $n \geq 1$  and all  $i \in \{0, n-1\}$ , the following:

$$T^i(0, ne_1) \text{ and } T(0, (n-1)e_1) \text{ have the same distribution.} \quad (1)$$

We now compare  $T^i(0, ne_1)$  and  $T(0, ne_1)$ . Let  $\pi$  be a path from 0 to  $ne_1$  such that  $\tau(\pi) = T(0, ne_1)$ . We modify this path as follows. Each time the path goes, in this order, through an edge  $(x, y) \in V^i$ , we replace this part of the path by  $(x, x + e_1, y + e_1, y)$ . We denote by  $\pi^i$  the modified path. We have

$$\tau^i(\pi^i) \leq \tau(\pi) - S_- \text{card}(\pi \cap H_i) + (S_+ - S_-) \text{card}(\pi \cap V_i)$$

where, for example,  $\text{card}(\pi \cap H_i)$  denotes the number of edges of  $H_i$  used by  $\pi$ . The term involving  $H_i$  is due to the time saved by the modification of the passage times. The term involving  $V_i$  is partly due to the time left by the modification of the path. We thus get

$$T^i(0, ne_1) \leq T(0, ne_1) - S_- \text{card}(\pi \cap H_i) + (S_+ - S_-) \text{card}(\pi \cap V_i)$$

and then

$$\sum_{i=0}^{n-1} T^i(0, ne_1) \leq nT(0, ne_1) - S_- \sum_{i=0}^{n-1} \text{card}(\pi \cap H_i) + (S_+ - S_-) \sum_{i=0}^{n-1} \text{card}(\pi \cap V_i). \quad (2)$$

Note

$$\sum_{i=0}^{n-1} \text{card}(\pi \cap H_i) \geq n,$$

as  $\pi$  is a path from 0 to  $ne_1$ . But

$$\begin{aligned} T(0, ne_1) &= \tau(\pi) \\ &\geq S_- \sum_{i=0}^{n-1} \text{card}(\pi \cap V_i) + S_- \sum_{i=0}^{n-1} \text{card}(\pi \cap H_i) \\ &\geq S_- \sum_{i=0}^{n-1} \text{card}(\pi \cap V_i) + S_- n \end{aligned}$$

and, moreover,

$$\begin{aligned} T(0, ne_1) &\leq \tau(0, e_1, 2e_2, \dots, n_1) \\ &\leq nS_+. \end{aligned}$$

Therefore:

$$\begin{aligned} \sum_{i=0}^{n-1} \text{card}(\pi \cap V_i) &\leq \frac{T(0, ne_1) - nS_-}{S_-} \\ &\leq \frac{nS_+ - nS_-}{S_-}. \end{aligned} \quad (3)$$

From (2) and (3) we get:

$$\sum_{i=0}^{n-1} T^i(0, ne_1) \leq nT(0, ne_1) - nS_- + \frac{n(S_+ - S_-)^2}{S_-}. \quad (4)$$

Taking expectations and using (1) we get:

$$na(n-1) \leq na(n) - nS_- \left[ 1 - \frac{(S_+ - S_-)^2}{S_-^2} \right].$$

The proof follows.

## References

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